Problem Set on Differentiation

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1. Let $f(x) = x$ if x is rational and $f(x) = \sin x$ if x is irrational. Find the points where f is differentiable.

2. If f is differentiable at every point of $[a, b]$, show that for every α such that $f'(a) < \alpha < f'(b)$, there exists $c \in (a, b)$ such that $f'(c) = \alpha$.

3. Prove that there is no value of k such that the equation $x^{2014} - 2014x + k = 0$ has two distinct roots in $[0, 1]$.

4. Let $f(x) = 0$ if $x \in [-1, 0]$ and $f(x) = 1$ if $x \in (0, 1]$. Does there exist a function g such that $g'(x) = f(x)$ for all $x \in [-1, 1]$?

5. Prove that $f(x) = \sin x$ is not a polynomial.

6. Is the map $\cos : \mathbb{R} \to \mathbb{R}$ a contraction ?. What about $\cos \circ \cos$?

7. Suppose $1 \leq f'(x) \leq 2$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all $x \in \mathbb{R}$.

8. If $f'(x) > 0$ for all x in an open interval I, then prove that f is injective on I. If f^{-1} a differentiable function ?

9. Show that the polynomial $x^{2n} - 2x^{2n-1} + 3x^{2n-2} + \cdots - 2nx + 2n + 1$ has no real root.

10. Show that the poynomial $\dfrac{x^n}{n!} + \dfrac{x^{n-1}}{(n-1)!} + \cdots + x + 1$ has no real root if n is even and exactly one real root if n is odd

11. Suppose that the polynomials P and Q have same roots, possibly with different multiplicities; and the same is true for $P + 1$ and $Q + 1$. Prove that $P = Q$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \dfrac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$. Show that f has a fixed point.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c)$.

14. Prove that the equation $e^x - ln(x) - 2^{2014} = 0$ has exactly two positive real roots.

15. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that

    (a) $f(x) \neq 0$ for all $x \in \mathbb{R}$;

    (b) $f(x) > 0$ for all $x \in \mathbb{R}$;

    (c) If f is differentiable at 0, then f is differentiable on $\mathbb{R}$ and there exists some real number β such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$.

16. Let f be a continuous function on $[0, 2]$ and twice differentiable on $(0, 2)$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then show that there exists x_0 such that $f''(x_0) = 0$.

17. For non-negative real numbers $a_1, a_2, \ldots a_n$, show that

$$\frac{1}{n}\sum_{k=1}^{n} a_k e^{-a_k} \leq \frac{1}{e}.$$

18. Let f be differentiable on $\mathbb{R}$ and $\sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Let $s_0 \in \mathbb{R}$. If $s_n = f(s_{n-1}$, prove that $\{s_n\}$ is convergent.

19. Let $f(x) = e^{-\frac{1}{x}}$ if $x > 0$, and $f(x) = 0$ if $x \leq 0$. Show that f is differentiable at 0 and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set $\{a \in \mathbb{R} : a$ is a local extremum of $f\}$ is countable.

21. Give an example of a real valued function of a real variable such that the set $\{a \in \mathbb{R} : a$ is a local extremum of $f\}$ is empty but the set $\{f(a) \in \mathbb{R} : f'(a) = 0\}$ is uncountable.